Synge's equations of motion applied to a perfect fluid

1979 J. Phys. A: Math. Gen. 12 503

(http://iopscience.iop.org/0305-4470/12/4/010)

J. Phys. A: Math. Gen., Vol. 12, No. 4, 1979. Printed in Great Britain

Synge's equations of motion applied to a perfect fluid

J D McCrea† and G O'Brien‡

† Department of Mathematical Physics, University College, Belfield, Dublin 4, Ireland

‡ School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Rd, Dublin 4, Ireland

Received 18 July 1978

Abstract. Conservation laws are derived using Synge's method in third approximation for a general continuous medium. These laws, together with Synge's equations of motion, are then applied to the case of a perfect fluid and a comparison is made with the results of Chandrasekhar and co-workers.

1. Introduction

An exhaustive treatment of a general relativistic perfect fluid has been given in a series of papers by Chandrasekhar and his co-workers (Chandrasekhar 1965, Chandrasekhar and Nutku 1969, Chandrasekhar and Esposito 1969). They employ a method of successive approximations based on expansions in inverse powers of the speed of light, carrying out the calculations to the $2\frac{1}{2}$-post-Newtonian approximation ($2\frac{1}{2}$-PNA) where the lowest-order radiation effects appear. The same problem has subsequently been treated by Anderson and Decanio (1975) using a different approach. Both approaches suffer from the disability of running into divergent integrals at the $2\frac{1}{2}$-PNA. Another method of successive approximations, but for a general continuous medium, is that due to Synge (1970). Synge showed that given stationary conditions at a finite time in the past, his method converges at all orders of approximation, and work at present in progress indicates that divergent integrals do not occur at the higher orders of approximation. In the present work, which is of a preliminary nature, we restrict ourselves to Synge's equations of motion in third approximation—corresponding more or less to Chandrasekhar's 2nd PNA—where radiation terms do not yet arise. In earlier papers (McCrea and O'Brien 1978, O'Brien 1978) these equations were applied to calculate the orbital and spin motion of a binary system and certain general results derivable from Synge's equations were indicated briefly. The purpose of the present work is to give a complete account of the general results underlying our calculations in our previous work and to show how these results compare with those of Chandrasekhar (1965) when applied to a perfect fluid.

In § 2 we derive the relations between two alternative decompositions of the energy tensor, T^{ab}, one being the exact decomposition in terms of the proper energy density, the four-velocity and the proper stress, the other being the approximate decomposition of our previous work (McCrea and O'Brien 1978, O'Brien 1978) which is a slightly modified version of that of Synge (1970). These relations will be used later on in making a comparison between our work and that of Chandrasekhar. In our previous work we assumed rigid motion so that the specific internal energy density, Π, was constant to the order of approximation considered and there was no need to make explicit mention of it. However, here we drop the rigid-motion condition and hence Π will enter explicitly into the equation of energy balance. This equation is derived in the context of the exact decomposition and then translated into the terms of the approximate decomposition of T^{ab}.

In § 3 we state Synge's equations of motion in third approximation for a general continuous medium and in § 4 the consequent conserved quantities are derived. In §§ 5, 6 and 7 the preceding work is applied to the case of a perfect fluid and the results for this case are found to be in agreement with those of Chandrasekhar (1965).

For ease of reference, we mention here the main points of notation which will frequently recur in what follows. Italic indices take the values (1, 2, 3, 4) and Greek indices the values (1, 2, 3). We use 'rectangular Cartesian' coordinates (x_1, x_2, x_3) and imaginary time $x_4 = \mathrm{i}t$, so that the metric has signature (+4).

For any function $f(\boldsymbol{x}, t)$, where $\boldsymbol{x} = (x_1, x_2, x_3)$, we have

$$D_\mu f = f_{,\mu} = \partial f/\partial x_\mu \qquad D_t f = f_{,t} = \partial f/\partial t \tag{1.1}$$

$$Jf(\boldsymbol{x}, t) = -(4\pi)^{-1} \int f(\boldsymbol{x}', t')|\boldsymbol{x} - \boldsymbol{x}'|^{-1}\, \mathrm{d}_3x' \tag{1.2}$$

where $t' = t - |\boldsymbol{x} - \boldsymbol{x}'|$;

$$I_n f(\boldsymbol{x}, t) = \int f(\boldsymbol{x}', t)|\boldsymbol{x} - \boldsymbol{x}'|^{n-1}\, \mathrm{d}_3x' \qquad (n = 0, 1, 2, \ldots); \tag{1.3}$$

along a world-line $x_\mu = x_\mu(t)$, $x_4 = \mathrm{i}t$,

$$\dot{f} = \mathrm{d}f/\mathrm{d}t = f_{,t} + f_{,\mu} v_\mu \tag{1.4}$$

where $v_\mu = \mathrm{d}x_\mu/\mathrm{d}t$.

Gravitational units are used throughout, so that both the gravitational constant and the velocity of light are unity.

2. The energy tensor for a general medium

The energy tensor, T^{ab}, of a continuous medium (cf Ehlers 1961) may be decomposed uniquely, with respect to any unit time-like vector u^a, in the form

$$T^{ab} = \mu u^a u^b + 2q^{(a}u^{b)} - \sigma^{ab} \tag{2.1}$$

where $\mu = T^{ab}u_a u_b$, $q^a = -\mu u^a - T^{ab}u_b$ and $q^a u_a = \sigma^{ab}u_b = 0$. In particular, if u^a is taken to be the local barycentric velocity of the medium (or average velocity of the 'particles'), then μ is the total proper energy density, q^a is the heat flow relative to u^a and σ^{ab} is the stress. In what follows, we shall consider only the case in which there is no heat flow (i.e. $q^a = 0$) so that

$$T^{ab} = \mu u^a u^b - \sigma^{ab} \tag{2.2}$$

and u^a is now the unit time-like eigenvector of T^{ab} with $-\mu$ as the corresponding eigenvalue.

If the expression (2.2) is substituted into the exact equations of motion,

$$T^{ab}{}_{|b} = 0 \tag{2.3}$$

and the resulting equations contracted with u_a, the equation of material energy balance

$$\mu_{\cdot a}u^a + \mu u^a{}_{|a} - \sigma^{ab}u_{a|b} = 0 \tag{2.4}$$

is obtained. It is assumed that the main part of the total proper energy density μ (Ehlers 1961) is due to the rest-mass density μ_0 which satisfies the conservation equation

$$(\mu_0 u^a)_{|a} = 0, \tag{2.5}$$

and the specific internal energy density Π is defined by

$$\mu = \mu_0(1+\Pi). \tag{2.6}$$

Substituting (2.6) and (2.5) into (2.4) we find that the material energy balance equation now takes the form

$$\mu_0\Pi_{\cdot a}u^a = \sigma^{ab}u_{a|b}. \tag{2.7}$$

So far everything has been exact. Let us now turn to the weak-field approximation considered by Synge (1970), McCrea and O'Brien (1978) and O'Brien (1978). To fix our ideas, we suppose that the universe with which we are dealing consists of a single body or several finite bodies of comparable mass and size and the small parameter, k, which forms the basis of the approximation is the mass/mean radius ratio of a typical body. We have

$$\begin{aligned} &\mu_0 = \iota^{-2}\mathrm{O}(k) \qquad &&\Pi = \mathrm{O}(k) \\ &u^\mu = \mathrm{O}(k^{1/2}) \qquad &&\sigma^{\mu\nu} = \iota^{-2}\mathrm{O}(k^2) \end{aligned} \tag{2.8}$$

where ι is a typical radius. In future we shall omit multiplication by the appropriate power of ι and simply write, for instance, $\mu_0 = \mathrm{O}(k)$. We also assume slow motion so that

$$\partial/\partial t = \mathrm{O}(k^{1/2}). \tag{2.9}$$

The decomposition (2.2) of the energy tensor implies the following alternative decomposition which is more convenient for our purposes:

$$T^{44} = -\rho \qquad T^{\mu 4} = \mathrm{i}(\rho v_\mu - S_{\mu\nu}v_\nu) + \mathrm{O}(k^{7/2}) \qquad T^{\mu\nu} = \rho v_\mu v_\nu - S_{\mu\nu} + \mathrm{O}(k^4) \tag{2.10}$$

where $v_\mu = \mathrm{i}u^\mu/u^4 = \mathrm{d}x_\mu/\mathrm{d}t$ along the world-lines $x_\mu(t)$ of the material particles, $\rho = -\mu(u^4)^2 - \sigma^{44}$ and $S_{\mu\nu} = \sigma^{\mu\nu}$.

The metric tensor in first approximation (cf Synge 1970, equation (9.5)) is

$$g_{ab} = \delta_{ab} + \gamma_{ab} \tag{2.11}$$

where

$$\gamma_{\mu\nu} = 2\delta_{\mu\nu}V + \mathrm{O}(k^2) \qquad \gamma_{\mu 4} = 4\mathrm{i}\,W_\mu + \mathrm{O}(k^{5/2}) \qquad \gamma_{44} = -2V + \mathrm{O}(k^2) \tag{2.12}$$

where

$$V = I_0(\rho) \qquad W_\mu = I_0(\rho v_\mu). \tag{2.13}$$

Using (2.11) and (2.12) one may verify that the relations between the quantities ρ, v_μ of

(2.10) and μ, u^a of (2.2) are given by

$$\begin{aligned} \mu &= \rho(1-v^2-2V)+O(k^3) \\ u^\mu &= (1+\tfrac{1}{2}v^2+V)v_\mu+O(k^{5/2}) \\ u^4 &= i(1+\tfrac{1}{2}v^2+V)+O(k^2) \\ u_\mu &= (1+\tfrac{1}{2}v^2+3V)v_\mu-4W_\mu+O(k^{5/2}) \\ u_4 &= i(1+\tfrac{1}{2}v^2-V)+O(k^2) \end{aligned} \tag{2.14}$$

where $v^2=v_\mu v_\mu$.

Substituting (2.13) into the exact material energy balance equation (2.7) and using (2.6), we obtain

$$\rho\, d\Pi/dt = S_{\mu\nu}v_{\mu'\nu}+O(k^{7/2}). \tag{2.15}$$

Equation (2.15) corresponds to the classical equation for isentropic motion.

3. The equations of motion for a general medium

In Synge's approximation method, starting from $T^{ab}=O(k)$ (in the sense that all components are less than or equal to $O(k)$), one generates a sequence of metrics

$$\underset{M}{g}_{ab}=\delta_{ab}+\underset{M}{\gamma}_{ab} \qquad (M=0,1,2,\ldots,N) \tag{3.1}$$

by the recurrence formula

$$\underset{0}{\gamma}_{ab}=0 \qquad \underset{M}{\gamma}^*_{ab}=-2\kappa J\underset{M-1}{H}^{ab} \qquad (M=1,2,\ldots,N) \tag{3.2}$$

where

$$\underset{M}{\gamma}^*_{ab}=\underset{M}{\gamma}_{ab}-\tfrac{1}{2}\delta_{ab}\underset{M}{\gamma}_{cc} \tag{3.3}$$

and

$$\underset{M}{H}^{ab}=T^{ab}+\kappa^{-1}\underset{M}{\hat{G}}^{ab} \qquad \kappa=8\pi. \tag{3.4}$$

Also

$$\underset{M}{\hat{G}}^{ab}=\underset{M}{G}^{ab}-\underset{M}{L}_{ab} \tag{3.5}$$

where $\underset{M}{G}^{ab}$ is the Einstein tensor for the metric $\underset{M}{g}_{ab}$ and

$$\underset{M}{L}_{ab}=\tfrac{1}{2}(\underset{M}{\gamma}_{ab,cc}+\underset{M}{\gamma}_{cc,ab}-\underset{M}{\gamma}_{ac,cb}-\underset{M}{\gamma}_{bc,ca})-\tfrac{1}{2}(\underset{M}{\gamma}_{cc,jj}+\underset{M}{\gamma}_{cj,cj})\delta_{ab}. \tag{3.6}$$

Equation (3.2) differs from Synge in the use of the J operator instead of Synge's more complicated K operator. However, use of the J operator leads to the same results.

If one stops at the Nth step and imposes on T^{ab} the equations

$$\underset{N-1}{H}^{ab}{}_{'b}=0 \tag{3.7}$$

then, as in Synge (1970), it can be shown that the $\underset{N}{g}_{ab}$ satisfy the equations

$$\underset{N}{G}^{ab} = -\kappa T^{ab} + \mathrm{O}(k^{N+1}) \tag{3.8}$$

so that the field equations are satisfied with an error of order k^{N+1}. It may also be shown that equations (3.7) imply

$$T^{ab}{}_{\underset{N}{|}b} = \mathrm{O}(k^{N+1}) \tag{3.9}$$

while conversely $T^{ab}{}_{\underset{N}{|}b} = 0$ imply $\underset{N-1}{H}^{ab}{}_{,b} = \mathrm{O}(k^{N+1})$, where the N under the vertical denotes covariant derivative with respect to $\underset{N}{g}_{ab}$. Hence, to order k^{N+1}, (3.7) are an equivalent form of the equations of motion.

In terms of the decomposition (2.10) of T^{ab}, Synge's equations of motion in third approximation ($N = 3$) are†

$$\begin{aligned} T^{\mu b}{}_{,b} &= \rho \dot{v}_\mu + (\dot{\rho} + \rho\theta) v_\mu - S_{\mu\nu,\nu} - (S_{\mu\nu} v_\nu)_{,t} \\ &= \rho V_{,\mu} + Y_\mu + \mathrm{O}(k^4) \end{aligned} \tag{3.10}$$

$$\begin{aligned} -\mathrm{i} T^{4b}{}_{,b} &= \dot{\rho} + \rho\theta - S_{\mu\nu,\nu} v_\mu - S_{\mu\nu} v_{\mu,\nu} \\ &= -\rho V_{,t} + \mathrm{O}(k^{7/2}) \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} Y_\mu = T^{\nu\nu} V_{,\mu} - 4 T^{\mu\nu} V_{,\nu} + 4\rho v_\nu (W_{\mu,\nu} - W_{\nu,\mu} - \delta_{\mu\nu} V_{,t}) \\ + \rho D_\mu [\tfrac{1}{2} D_t^2 (I_2 \rho) - 2 V^2 + K_{\nu\nu}] + 4\rho D_t W_\mu = \mathrm{O}(k^3) \end{aligned} \tag{3.12}$$

and

$$K_{\mu\nu} = I_0 (T^{\mu\nu}) \qquad \theta = v_{\mu,\mu}. \tag{3.13}$$

On multiplication by v_μ, equation (3.10) yields

$$S_{\mu\nu,\nu} v_\mu = \rho \frac{\mathrm{d}}{\mathrm{d}t} (\tfrac{1}{2} v^2) - \rho V_{,\mu} v_\mu + \mathrm{O}(k^{7/2}). \tag{3.14}$$

Substituting (3.14) and (2.15) into (3.11) we then obtain

$$\frac{\mathrm{d}\rho^*}{\mathrm{d}t} + \rho^* \theta = \mathrm{O}(k^{7/2}) \tag{3.15}$$

where

$$\rho^* = \rho (1 - \tfrac{1}{2} v^2 + V - \Pi). \tag{3.16}$$

ρ^* is the familiar post-Newtonian conserved density.

4. The conserved quantities for a general medium

Synge's equations of motion in Nth approximation in the form given by (3.7) imply the

† Cf McCrea and O'Brien (1978), equations (2.23) and (2.24).

conservation of the quantities

$$P_\mu = -\mathrm{i}\int \underset{N-1}{H}^{\mu 4}\ \mathrm{d}_3x \tag{4.1}$$

$$P_4 = -\mathrm{i}\int \underset{N-1}{H}^{44}\ \mathrm{d}_3x \tag{4.2}$$

and

$$P_{\mu\nu} = -\mathrm{i}\int [x_\mu \underset{N-1}{H}^{\nu 4} - x_\nu \underset{N-1}{H}^{\mu 4}]\,\mathrm{d}_3x \tag{4.3}$$

where the integrals are taken over all of space. We refer to P_μ, $-\mathrm{i}P_4$ and $P_{\mu\nu}$ as the total three-momentum, mass–energy and angular momentum of the universe respectively. Since the supports of the integrands above are non-compact the question of convergence of the integrals (4.1)–(4.3) inevitably arises. We are assuming that the system was stationary at some finite time in the past and hence, as shown by Synge (1970), $\underset{N-1}{\hat{G}}{}^{ab}$ goes to zero like r^{-4} as $r \to \infty$. Therefore in this case, since T^{ab} is assumed to have compact support, the integrals (4.1) and (4.2) certainly converge. However, even in this case, the convergence of (4.3) would remain doubtful. In the present work we do not intend to treat the convergence of (4.3) in full generality. Consideration is restricted to the third approximation ($N = 3$) with $\underset{2}{H}^{\mu 4}$ and $\underset{2}{H}^{44}$ evaluated explicitly to order $k^{5/2}$ and k^2 respectively.

In what follows we shall drop the suffix $N-1$ (with $N = 3$) in the integrals (4.1)–(4.3). Also, for the sake of brevity, we shall describe as compact any function whose restriction to a hypersurface $t = \text{constant}$ has compact support.

From (2.10), (3.4) and equation (1.51) of Synge (1970) we have

$$H^{\mu 4} = \mathrm{i}(\rho v_\mu - S_{\mu\sigma}v_\sigma) - \frac{3\mathrm{i}}{4\pi} V_{\cdot\mu} V_{\cdot 4} + \frac{\mathrm{i}}{4\pi}(2V_{\cdot\sigma}W_{\sigma\cdot\mu} - 2V_{\cdot\mu\sigma}W_\sigma - 2VW_{\mu\cdot\sigma\sigma} + 2W_\mu V_{\cdot\sigma\sigma}) + \mathrm{O}(k^{7/2}) \tag{4.4}$$

where it should be noted that the instantaneous potentials $\tilde{V}$, $\tilde{W}_\mu$ and $\tilde{K}_{\mu\nu}$ of Synge's paper are denoted simply by V, W_μ and $K_{\mu\nu}$ in the present work. We shall now show that the non-compact integrals in (4.1) and (4.3) may be replaced by compact ones. To do this we resolve the integrands into two parts—a compact part and a non-compact part which gives no contribution to the integrals. Let us take the integral (4.1) first.

From (2.13) we have

$$W_{\mu\cdot\nu\nu} = -4\pi\rho v_\mu \tag{4.5}$$

and

$$V_{\cdot\nu\nu} = -4\pi\rho. \tag{4.6}$$

Also, by equation (3.11), it is easily verified that

$$V_{\cdot 4} = \mathrm{i}W_{\nu\cdot\nu} + \mathrm{O}(k^{5/2}). \tag{4.7}$$

Using (4.5)–(4.7), we may write (4.4) in the form

$$H^{\mu 4} = H'^{\mu 4} + H''^{\mu 4} + \mathrm{O}(k^{7/2}) \tag{4.8}$$

where

$$H'^{\mu 4} = \mathrm{i}[\rho v_\mu - S_{\mu\sigma} v_\sigma + 4\rho (V v_\mu - W_\mu)] \tag{4.9}$$

and

$$H''^{\mu 4} = \frac{\mathrm{i}}{4\pi}(-3 V_{\cdot\mu} W_{\sigma\cdot\sigma} + 2 V_{\cdot\sigma} W_{\sigma\cdot\mu} - 2 V_{\cdot\mu\sigma} W_\sigma + 2 V W_{\mu\cdot\sigma\sigma} - 2 W_\mu V_{\cdot\sigma\sigma}). \tag{4.10}$$

$H'^{\mu 4}$ is compact while $H''^{\mu 4}$ is not. For large r, where $r = |\boldsymbol{x}|$ and the origin of coordinates is within the (compact) support of ρ, V and W_μ fall off like r^{-1} and hence $H''^{\mu 4}$ falls off like r^{-4}. Equation (4.10) may be written in the alternative form

$$H''^{\mu 4} = \frac{\mathrm{i}}{2\pi}[(V W_{\sigma\cdot\mu} + V W_{\mu\cdot\sigma} - V_{\cdot\mu} W_\sigma - V_{\cdot\sigma} W_\mu)_{\cdot\sigma} - (V W_{\sigma\cdot\sigma})_{\cdot\mu}] + \frac{\mathrm{i}}{4\pi}(V_{\cdot\mu} W_{\sigma\cdot\sigma}), \tag{4.11}$$

and, since the first term of (4.11) is the divergence of a quantity which falls off like r^{-3} at great distances, its volume integral vanishes by Gauss' theorem. Hence

$$\int H''^{\mu 4}\, \mathrm{d}_3 x = \frac{\mathrm{i}}{4\pi} \int V_{\cdot\mu} W_{\sigma\cdot\sigma}\, \mathrm{d}_3 x. \tag{4.12}$$

Thus

$$H''^{\mu 4} = \frac{\mathrm{i}}{4\pi} V_{\cdot\mu} W_{\sigma\cdot\sigma} + H'''^{\mu 4} \tag{4.13}$$

where

$$\int H'''^{\mu 4}\, \mathrm{d}_3 x = 0. \tag{4.14}$$

$H''^{\mu 4}$ is still non-compact but it may be resolved further as follows. It is shown in the Appendix that

$$\frac{\mathrm{i}}{4\pi} V_{\cdot\mu} W_{\sigma\cdot\sigma} = \tfrac{1}{2}\mathrm{i}\rho D_{\mu\sigma} I_2(\rho v_\sigma) + \frac{\mathrm{i}}{4\pi}(V W_{\sigma\cdot\sigma})_{\cdot\mu} + \frac{\mathrm{i}}{8\pi}[V_{\cdot\sigma} D_{\mu\nu} I_2(\rho v_\nu) - V D_{\mu\nu\sigma} I_2(\rho v_\nu)]_{\cdot\sigma}. \tag{4.15}$$

Again, by Gauss' theorem, the volume integral of the last two terms of (4.15) vanishes and hence

$$\frac{\mathrm{i}}{4\pi} \int V_{\cdot\mu} W_{\sigma\cdot\sigma}\, \mathrm{d}_3 x = \frac{\mathrm{i}}{2} \int \rho D_{\mu\sigma} I_2(\rho v_\sigma)\, \mathrm{d}_3 x. \tag{4.16}$$

Therefore, by (4.8), (4.9), (4.12) and (4.16) the integral (4.1) may be written in the form

$$P_\mu = \int H_\mu\, \mathrm{d}_3 x + \mathrm{O}(k^{7/2}) \tag{4.17}$$

where

$$H_\mu = \rho v_\mu - S_{\mu\sigma} v_\sigma + \tfrac{1}{2}\rho D_{\mu\sigma} I_2(\rho v_\sigma) + 4\rho (V v_\mu - W_\mu). \tag{4.18}$$

A similar procedure can be used for the integral (4.3). However, since the integrand in this case falls off like r^{-3}, instead of r^{-4}, one must be more careful in applying Gauss' theorem. First of all, one writes $H^{\mu 4}$ in the form given by (4.8)–(4.10). With $H''^{\mu 4}$ as in

(4.10) one may write, after some straightforward calculation,

$$
\begin{aligned}
\int (x_\mu H''^{\nu 4} - x_\nu H''^{\mu 4})\, \mathrm{d}_3 x
&= \frac{\mathrm{i}}{4\pi} \int (x_\mu V_{\cdot\nu} - x_\nu V_{\cdot\mu}) W_{\sigma\cdot\sigma}\, \mathrm{d}_3 x \\
&\quad + \frac{\mathrm{i}}{2\pi} \int [x_\mu (V W_{\sigma\cdot\nu} + V W_{\nu\cdot\sigma} - V_{\cdot\nu} W_\sigma - V_{\cdot\sigma} W_\nu)]_{\cdot\sigma}\, \mathrm{d}_3 x \\
&\quad - \frac{\mathrm{i}}{2\pi} \int [x_\nu (V W_{\sigma\cdot\mu} + V W_{\mu\cdot\sigma} - V_{\cdot\mu} W_\sigma - V_{\cdot\sigma} W_\mu)]_{\cdot\sigma}\, \mathrm{d}_3 x \\
&\quad - \frac{\mathrm{i}}{2\pi} \int (x_\mu V W_{\sigma\cdot\sigma})_{\cdot\nu}\, \mathrm{d}_3 x + \frac{\mathrm{i}}{2\pi} \int (x_\nu V W_{\sigma\cdot\sigma})_{\cdot\mu}\, \mathrm{d}_3 x.
\end{aligned} \tag{4.19}
$$

Application of Gauss' theorem to the last four integrals of (4.19) then yields

$$
\begin{aligned}
\int (x_\mu H''^{\nu 4} - x_\nu H''^{\mu 4})\, \mathrm{d}_3 x
&= \frac{\mathrm{i}}{4\pi} \int (x_\mu V_{\cdot\nu} - x_\nu V_{\cdot\mu}) W_{\sigma\cdot\sigma}\, \mathrm{d}_3 x \\
&\quad + \frac{\mathrm{i}}{2\pi} \int_S x_\mu (V W_{\sigma\cdot\nu} + V W_{\nu\cdot\sigma} - V_{\cdot\nu} W_\sigma - V_{\cdot\sigma} W_\nu) n_\sigma\, \mathrm{d}S \\
&\quad - \frac{\mathrm{i}}{2\pi} \int_S x_\nu (V W_{\sigma\cdot\mu} + V W_{\mu\cdot\sigma} - V_{\cdot\mu} W_\sigma - V_{\cdot\sigma} W_\mu) n_\sigma\, \mathrm{d}S \\
&\quad - \frac{\mathrm{i}}{2\pi} \int_S x_\mu V W_{\sigma\cdot\sigma} n_\nu\, \mathrm{d}S + \frac{\mathrm{i}}{2\pi} \int_S x_\nu V W_{\sigma\cdot\sigma} n_\mu\, \mathrm{d}S
\end{aligned} \tag{4.20}
$$

where the surface of integration S is the sphere at infinity and n_σ is the unit normal to S. However, the integrands in the surface integrals fall off only like r^{-2} so that, in order to see whether they vanish, we must examine them in more detail.

At great distances

$$V = mr^{-1} + \mathrm{O}(r^{-2}) \tag{4.21}$$

where

$$m = \int \rho(\boldsymbol{x}', t)\, \mathrm{d}_3 x' \tag{4.22}$$

and

$$W_\sigma = \phi_\sigma r^{-1} + \mathrm{O}(r^{-2}) \tag{4.23}$$

where

$$\phi_\sigma = \int \rho(\boldsymbol{x}', t) v_\sigma(\boldsymbol{x}', t)\, \mathrm{d}_3 x'. \tag{4.24}$$

Also, on S,

$$n_\sigma = x_\sigma r^{-1}. \tag{4.25}$$

Furthermore, for a sphere S of radius r,

$$\int_S f(r)x_\mu x_\nu x_\sigma \, dS = 0. \tag{4.26}$$

Hence, using (4.21)–(4.26), it is seen that the surface integrals in (4.20) all vanish, so that

$$\int (x_\mu H''^{\nu 4} - x_\nu H''^{\mu 4}) \, d_3x = \frac{i}{4\pi}\int (x_\mu V_{\cdot\nu} - x_\nu V_{\cdot\mu}) W_{\sigma\cdot\sigma} \, d_3x. \tag{4.27}$$

If we now substitute (4.15) into (4.27), some straightforward calculations, and use of the same arguments as before for the surface integrals obtained by applying Gauss' theorem, yield the result

$$\frac{i}{4\pi}\int (x_\mu V_{\cdot\nu} - x_\nu V_{\cdot\mu}) W_{\sigma\cdot\sigma} \, d_3x = \frac{i}{2}\int \rho(x_\mu D_{\nu\sigma} - x_\nu D_{\mu\sigma}) I_2(\rho v_\sigma) \, d_3x. \tag{4.28}$$

Thus, combining the equations (4.8), (4.9), (4.27) and (4.28) we obtain for the integral (4.3)

$$P_{\mu\nu} = \int (x_\mu H_\nu - x_\nu H_\mu) \, d_3x + O(k^{7/2}) \tag{4.29}$$

where H_μ is given by (4.18).

Finally, for the integral (4.2) the procedure is much simpler. By (2.10), (3.4) and equation (1.51) of Synge (1970)

$$H^{44} = -\rho(1+2V) + \frac{3}{8\pi} V_{\cdot\sigma} V_{\cdot\sigma} + O(k^3), \tag{4.30}$$

and, by straightforward application of Gauss' theorem, this yields

$$P_4 = i\int H_4 \, d_3x + O(k^3) \tag{4.31}$$

where

$$H_4 = \rho(1 + \tfrac{1}{2}V). \tag{4.32}$$

Thus, in equations (4.17), (4.29) and (4.32), we have succeeded in our aim of expressing P_μ, P_4 and $P_{\mu\nu}$ as integrals with compact integrands, to the order in k indicated.

In addition to (4.1)–(4.3), Synge's equations of motion imply the conservation of

$$\begin{aligned} P_{\mu 4} &= -i\int (x_\mu H^{44} - x_4 H^{\mu 4}) \, d_3x \\ &= \int x_\mu H_4 \, d_3x - x_4 \int H_\mu \, d_3x + O(k^3) \end{aligned} \tag{4.33}$$

from which the definition and motion of the centre of mass of the system may be derived, as in § 3 of McCrea and O'Brien (1978).

5. Energy tensor for a perfect fluid

In this and in the following sections we apply the foregoing results to the case of a perfect fluid in order to make a comparison with the work of Chandrasekhar (1965). For a perfect fluid the stress tensor σ^{ab} is given by

$$\sigma^{ab} = -p(g^{ab} + u^a u^b) \tag{5.1}$$

where p is the isotropic pressure, and the decomposition (2.2) then takes the form

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab}. \tag{5.2}$$

We recall from (2.6) that μ may be written

$$\mu = \mu_0(1 + \Pi). \tag{5.3}$$

Note that our μ and μ_0 correspond to Chandrasekhar's ϵ and ρ respectively, while our ρ will retain the same meaning it had in the previous sections, namely $\rho = -T^{44}$.

In order to parallel Chandrasekhar's developments, we define the following functionals of μ_0 and v_μ:

$$U = I_0(\mu_0) = \mathrm{O}(k) \tag{5.4}$$

and

$$U_\mu = I_0(\mu_0 v_\mu) = \mathrm{O}(k^{3/2}). \tag{5.5}$$

Using (2.13), (2.14) and (5.3) one may verify that

$$U = V + \mathrm{O}(k^2) \tag{5.6}$$

and

$$U_\mu = W_\mu + \mathrm{O}(k^{5/2}). \tag{5.7}$$

If we now combine (2.14) with (5.3), (5.6) and (5.7) we obtain

$$\rho = \mu_0(1 + v^2 + 2U + \Pi) + \mathrm{O}(k^3) \tag{5.8}$$

$$u^\mu = (1 + \tfrac{1}{2}v^2 + U)v_\mu + \mathrm{O}(k^{5/2}) \tag{5.9}$$

$$u^4 = \mathrm{i}(1 + \tfrac{1}{2}v^2 + U) + \mathrm{O}(k^2) \tag{5.10}$$

$$u_\mu = (1 + \tfrac{1}{2}v^2 + 3U)v_\mu - 4U_\mu + \mathrm{O}(k^{5/2}) \tag{5.11}$$

$$u_4 = \mathrm{i}(1 + \tfrac{1}{2}v^2 - U) + \mathrm{O}(k^2), \tag{5.12}$$

and on comparing the last four expressions with equations (15) and (16) of Chandrasekhar (1965) we see that they agree.

Substituting (5.8) into the decomposition (2.10) of T^{ab} we may write the contravariant components of the energy tensor as

$$T^{\mu\nu} = \mu_0(1 + v^2 + 2U + \Pi)v_\mu v_\nu - S_{\mu\nu} + \mathrm{O}(k^4) \tag{5.13}$$

$$T^{4\mu} = \mathrm{i}[\mu_0(1 + v^2 + 2U + \Pi)v_\mu - S_{\mu\nu}v_\nu] + \mathrm{O}(k^{7/2}) \tag{5.14}$$

$$T^{44} = -\mu_0(1 + v^2 + 2U + \Pi) + \mathrm{O}(k^3). \tag{5.15}$$

To find $S_{\mu\nu}$ for a perfect fluid to $\mathrm{O}(k^3)$, we recall that, by definition, $S_{\mu\nu} = \sigma^{\mu\nu}$ and hence, by (5.1),

$$S_{\mu\nu} = -p(u^\mu u^\nu + g^{\mu\nu}). \tag{5.16}$$

Therefore, by (5.9) and the fact that

$$g^{\mu\nu} = \delta_{\mu\nu}(1 - 2U + O(k^2)), \tag{5.17}$$

we obtain

$$S_{\mu\nu} = -p\delta_{\mu\nu} + p(2U\delta_{\mu\nu} - v_\mu v_\nu) + O(k^4). \tag{5.18}$$

If (5.18) is then substituted into (5.13)–(5.15) one obtains the energy tensor in terms of μ_0, v_μ, Π and p in the form

$$T^{\mu\nu} = \mu_0 v_\mu v_\nu(1 + v^2 + 2U + \Pi + p/\mu_0) + p(1 - 2U)\delta_{\mu\nu} + O(k^4) \tag{5.19}$$

$$T^{4\mu} = i\mu_0(1 + v^2 + 2U + \Pi + p/\mu_0)v_\mu + O(k^{7/2}) \tag{5.20}$$

$$T^{44} = -\mu_0(1 + v^2 + 2U + \Pi) + O(k^3), \tag{5.21}$$

in agreement with equations (20) of Chandrasekhar (1965).

6. The equations of motion for a perfect fluid

Synge's equations of motion in third approximation read

$$T^{\mu b}{}_{,b} = \rho V_{,\mu} + Y_\mu + O(k^4) \tag{6.1}$$

$$-iT^{4b}{}_{,b} = -\rho V_{,t} + O(k^{7/2}) \tag{6.2}$$

where Y_μ is given by (3.12). By substituting the components (5.19)–(5.21) of the energy tensor into the left-hand sides of (6.1) and (6.2) we obtain the latter in terms of μ_0, v_μ, Π and p. In order to effect a comparison with Chandrasekhar's equations of motion we must also find the right-hand sides in terms of these variables. To do this, we first of all note that, by (2.13), (5.4) and (5.8),

$$V = U + I_0[\mu_0(v^2 + 2U + \Pi)] + O(k^3) \tag{6.3}$$

and, by (3.13) and (5.19),

$$K_{\sigma\sigma} = I_0(\mu v^2 + 3p) + O(k^3). \tag{6.4}$$

Chandrasekhar uses two further functions, Φ and X, defined by means of the equations

$$\nabla^2\Phi = -4\pi\mu_0(v^2 + U + \tfrac{1}{2}\Pi + 3p/2\mu_0) \tag{6.5}$$

and

$$\nabla^2 X = -2U, \tag{6.6}$$

or, equivalently, Φ and X are given by the integral representations

$$\Phi = I_0[\mu_0(v^2 + U + \tfrac{1}{2}\Pi + 3p/2\mu_0)] \tag{6.7}$$

and

$$X = -I_2\mu_0. \tag{6.8}$$

He also defines a function σ by

$$\sigma = \mu_0(1 + v^2 + 2U + \Pi + p/\mu_0). \tag{6.9}$$

If we now substitute (5.19)–(5.21) into the left-hand side of (6.1), with (6.3), (6.4), (6.7) and (6.8) in the right-hand side, and use (6.9), we obtain, after some straightforward algebra,

$$\begin{aligned} D_\nu(\sigma v_\mu v_\nu) + D_t(\sigma v_\mu) + (1-2U)p_{\cdot\mu} - \sigma(1+v^2)U_{\cdot\mu} + 4\mu_0 v_\mu(U_{\cdot t} + v_\nu U_{\cdot\nu}) \\ -4\mu_0[U_{\mu\cdot t} + v_\nu(U_{\mu\cdot\nu} - U_{\nu\cdot\mu})] + \mu_0 D_\mu(2U^2 - 2\Phi + \tfrac{1}{2}D_t^2 X) \\ = O(k^4), \end{aligned} \tag{6.10}$$

in agreement with equation (67) of Chandrasekhar ((1965).

Equation (6.2) is treated somewhat more simply. It may be written

$$D_\mu[\mu_0(1+v^2+2U+\Pi+p/\mu_0)v_\mu] + D_t[\mu_0(1+v^2+2U+\Pi)] = -\mu_0 U_{\cdot t} + O(k^{7/2}), \tag{6.11}$$

which, on substituting for σ from (6.9), yields

$$D_\mu(\sigma v_\mu) + D_t\sigma + \mu_0 U_{\cdot t} - p_{\cdot t} = O(k^{7/2}), \tag{6.12}$$

which is the same as equation (64) of Chandrasekhar (1965). Hence, Synge's equations of motion in third approximation for a perfect fluid agree with Chandrasekhar's first post-Newtonian equations of hydrodynamics.

7. The conserved quantities for a perfect fluid

Let us first consider the linear three-momentum and the angular momentum. Chandrasekhar has shown that the post-Newtonian equations of motion for a perfect fluid allow one to define a conserved linear momentum

$$\int \pi_\mu \, d_3x = \text{constant} \tag{7.1}$$

and a conserved angular momentum

$$\int (x_\mu \pi_\nu - x_\nu \pi_\mu) \, d_3x = \text{constant} \tag{7.2}$$

where

$$\pi_\mu = \sigma v_\mu + \tfrac{1}{2}\mu_0(U_\mu - U_{\nu;\mu\nu}) + 4\mu_0(v_\mu U - U_\mu) \tag{7.3}$$

and

$$U_{\mu;\nu\sigma} = \int \frac{\mu_0(\boldsymbol{x}', t)v_\mu(\boldsymbol{x}', t)(x_\nu - x'_\nu)(x_\sigma - x'_\sigma)}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \, d_3x'. \tag{7.4}$$

We shall now show that H_μ as given by (4.18) agrees with π_μ for the case of a perfect fluid and to the order of approximation considered.

For a perfect fluid

$$H_\mu = \rho v_\mu + p v_\mu + \tfrac{1}{2}\rho D_{\mu\nu} I_2(\rho v_\nu) + 4\rho(V v_\mu - W_\mu) + O(k^{7/2}). \tag{7.5}$$

Substitution of (5.6)–(5.8) into (7.5) yields

$$H_\mu = \mu_0(1+v^2+2U+\Pi+p/\mu_0)v_\mu + \tfrac{1}{2}\mu_0 D_{\mu\nu} I_2(\mu_0 v_\nu) + 4\mu_0(U v_\mu - U_\mu) + O(k^{7/2}). \tag{7.6}$$

From the definition (7.4) of $U_{\mu;\nu\sigma}$, it is easy to verify that

$$D_{\mu\nu} I_2(\mu_0 v_\nu) = U_\mu - U_{\nu;\mu\nu}. \tag{7.7}$$

Hence, combining (6.9) with (7.6) and (7.7), we obtain

$$H_\mu = \sigma v_\mu + \tfrac{1}{2}\mu_0(U_\mu - U_{\nu;\mu\nu}) + 4\mu_0(Uv_\mu - U_\mu) + \mathrm{O}(k^{7/2}) \qquad (7.8)$$

for a perfect fluid, in agreement with π_μ as defined by equation (128) of Chandrasekhar (1965).

The so called post-Newtonian rest-mass density ρ^* was defined by (3.16) and satisfies the continuity equation (3.15), so that the rest mass m_0 of the system, defined by

$$m_0 = \int \rho^* \, \mathrm{d}_3 x, \qquad (7.9)$$

satisfies

$$\mathrm{d}m_0/\mathrm{d}t = \mathrm{O}(k^{7/2}). \qquad (7.10)$$

An alternative expression for ρ^* is obtained by substituting (5.8) into (3.16) to obtain

$$\rho^* = \mu_0(1 + \tfrac{1}{2}v + 3U), \qquad (7.11)$$

as in equation (118) of Chandrasekhar (1965). In § 4 we showed that the fourth component, P_4, of the conserved total four-momentum is expressible in the form given by (4.31) and (4.32). Defining the total mass–energy, m, of the system by

$$m = -\mathrm{i}P_4, \qquad (7.12)$$

and expressing ρ in (4.32) in terms of ρ^* we obtain

$$m = \int \rho^*(1 + \tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi) \, \mathrm{d}_3 x + \mathrm{O}(k^3). \qquad (7.13)$$

If we then define the total energy of the fluid system by

$$E = m - m_0, \qquad (7.14)$$

then, by (7.9) and (7.13),

$$E = \int \rho^*(\tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi) \, \mathrm{d}_3 x + \mathrm{O}(k^3) \qquad (7.15)$$

and

$$\mathrm{d}E/\mathrm{d}t = \mathrm{O}(k^{7/2}). \qquad (7.16)$$

The explicit term of (7.15) is simply the total energy for a Newtonian self-gravitating system.

The quantity $P_{\mu 4}$ defined by (4.33) is also conserved as a consequence of Synge's equations of motion. For a perfect fluid the conservation of $P_{\mu 4}$ leads to an equation for centre-of-mass motion which agrees with that derived by Contopoulos and Spyrou (1976, see equations (17) and (18)) using Chandrasekhar's equations of hydrodynamics.

8. Conclusion

We have not considered the metric in the above. A comparison between the metric derived on the basis of Synge's equations and that derived by Chandrasekhar and Nutku (1969) in the second PNA shows that they agree, to order k^2 in the case of $g_{\mu\nu}$ and g_{44} and to order $k^{5/2}$ in the case of $g_{4\mu}$, up to a gauge transformation, similar to but not the same as that derived by Anderson and Decanio (1975). Work now in progress indicates that the divergent integrals which occur in the work of Anderson and Decanio (1975) and Chandrasekhar and Esposito (1969) do not arise in Synge's gauge. One further approximation in Synge's scheme will lead to the lowest-order radiation terms in parallel with the work of Chandrasekhar and Esposito (1969) and that of Anderson and Decanio (1975). This step is now being calculated and a procedure along the lines of that used by McCrea and O'Brien (1978), O'Brien (1978) and Hogan and McCrea (1974) should enable one to calculate the radiation damping for a binary system.

Appendix

To show that

$$\frac{\mathrm{i}}{4\pi}V_{\cdot\mu}W_{\nu\cdot\nu}=\frac{\mathrm{i}}{2}\rho D_{\mu\nu}I_2(\rho v_\nu)+\frac{\mathrm{i}}{4\pi}(VW_{\nu\cdot\nu})_{\cdot\mu}+\frac{\mathrm{i}}{8\pi}[V_{\cdot\sigma}D_{\mu\nu}I_2(\rho v_\nu)-VD_{\mu\nu\sigma}I_2(\rho v_\nu)]_{\cdot\sigma}: \tag{A.1}$$

By (1.3)

$$I_2(\rho v_\mu)=\int \rho' v'_\mu|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}_3x', \tag{A.2}$$

and differentiating this one obtains

$$2W_\mu=D_{\nu\nu}I_2(\rho v_\mu). \tag{A.3}$$

Also, one may write

$$\frac{1}{2}\rho D_{\mu\nu}I_2(\rho v_\nu)=-\frac{1}{8\pi}V_{\cdot\sigma\sigma}D_{\mu\nu}I_2(\rho v_\nu), \tag{A.4}$$

or, after some manipulation of the right-hand side of (A.4)

$$\begin{aligned}\frac{1}{2}\rho D_{\mu\nu}I_2(\rho v_\nu)&\\ &=-\frac{1}{8\pi}[VD_{\nu\sigma\sigma}I_2(\rho v_\nu)]_{\cdot\mu}+\frac{1}{8\pi}[VD_{\mu\nu\sigma}I_2(\rho v_\nu)-V_{\cdot\sigma}D_{\mu\nu}I_2(\rho v_\nu)]_{\cdot\sigma}\\ &\quad+\frac{1}{8\pi}V_{\cdot\mu}D_{\nu\sigma\sigma}I_2(\rho v_\nu).\end{aligned} \tag{A.5}$$

Combining (A.3) and (A.5), the result (A.1) follows.

References

Anderson J L and Decanio T C 1975 *Gen. Rel. Gravit.* **6** 197
Chandrasekhar S 1965 *Astrophys. J.* **142** 1488
Chandrasekhar S and Esposito F P 1969 *Astrophys. J.* **160** 153
Chandrasekhar S and Nutku Y 1969 *Astrophys. J* **158** 55
Contopoulos G and Spyrou N 1976 *Astrophys. J.* **205** 592
Ehlers J 1961 *Abh. Akad. Wiss. und Lit. Mainz, Math. Nat.* Kl no 11
Hogan P A and McCrea J D 1974 *Gen. Rel. Gravit.* **5** 79
McCrea J D and O'Brien G 1978 *Gen. Rel. Gravit.* in the press.
O'Brien G 1978 *Gen. Rel. Gravit.* in the press.
Synge J L 1970 *Proc. R. Ir. Acad.* A **69** 11